Basic Probability Theory

- Outcome – Set of possible Outcomes
- Event – Subset of possible Outcomes
- Distribution – measure on the outcome space
- Independence – "Events that are unrelated".

Discrete Examples:

- Coin Flip:
  - Outcomes: $\{H, T\}$
  - Events: $\{\{\}, \{H\}, \{T\}, \{H, T\}\}$
  - Distribution: $p(\{H\}) = p(\{T\}) = 0.5$

Just give the probability of each outcom $p(\{H\}) = 0.2,\ p(\{T\}) = 0.8$

- Rolling a Die:
  - Outcomes: $\{1, 2, 3, 4, 5, 6\}$
  - Events: $\{3\}$ – rolling a 3

" Something you can bet on"

$E_1 = \{1, 3, 5\}$ – roll an odd number

$\{2, 4, 6\}$ – roll an even number

$E_2 = \{1, 2\}$ – roll either a 1 or 2.

An event is something which can happen or not happen.

  - Distribution: $p(\{1\}) = p(\{2\}) = \cdots = p(\{6\}) = \frac{1}{6}$

$p(\{1\}) = \frac{1}{2},\ p(\{2\}) = \cdots = p(\{6\}) = \frac{1}{10}$

  - Independence: Given events E_1 and E_2, they are independent if

Depends on the distribution

$$p(E_1 \cap E_2) = p(E_1)\,p(E_2)$$

$$p(E_1 | E_2) = \frac{p(E_1 \cap E_2)}{p(E_2)} = p(E_1)$$

$E_1 \cap E_2 = \{1, 3, 5\} \cap \{1, 2\} = \{1\}$

$p(E_1 \cap E_2) = p(\{1\}) = \frac{1}{6}$

$p(E_1)\,p(E_2) = p(\{1,3,5\})\,p(\{1,2\}) =$

$= \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

~~even~~ ~~odd~~ ~~the die roll~~ mod 2 and mod 3 are independent.

S. $p(E_1 \cap E_2) = p(E_1)\,p(E_2) \Rightarrow E_1$ and E_2 independent.

Independent Copies:

May not be true given a different distribution

- Coin Flip: 2 independent copies:

  $C_1 = \{H, T\}$
  $C_2 = \{H, T\}$ $\Rightarrow C_1 \times C_2$

	H^{p_H}	T^{p_T}
p_H H	HH	HT
p_T T	TH	TT

Events: $\{HT, TH\}$ – exactly one head

$\{HH, HT, TH\}$ – at least one head

- Distribution: $p((H,T)) = p_1(H)\,p_2(T)$

distribution for first coin

dist. for second coin.

need not be the same.

- Generalized: k-coins indep.

  $\{H,T\}^k = \{k\text{-tuples of } H, T\}$

Dist. $p((H, \dots, T)) = p_1(H) \cdots p_k(T)$

Can generalize this to infinite products as well

Continuous Distributions

Ex: Uniform Distribution on $[0,1]$.

Outcomes: $[0,1]$

Events: Ex: $[0,.5]$ random number ≤ 0.5. ← First bit $=0$

Ex: $[0,\frac{1}{4}]\cup[\frac{1}{2},\frac{3}{4}]$, second digit in binary expansion is 0.

Second bit $=0$

Distribution: A distribution a rule which gives the probability of any event. Properties to satisfy:

- $E_1, E_2, \ldots E_n, \ldots$ s.t. $E_i \cap E_j = \phi$,

then $p(E_1 \cup \ldots \cup E_n \cup \ldots) = \sum_{i=1}^{\infty} p(E_i)$. countable additivity

$$p(E) = \int_E dx \quad \text{"length of } E\text{"}.$$

check this satisfies the above property →

$$p(E_1) = \int_{E_1} dx = \int_0^{1/2} dx = \frac{1}{2}, \quad p(E_2) = \int_{E_2} dx =$$

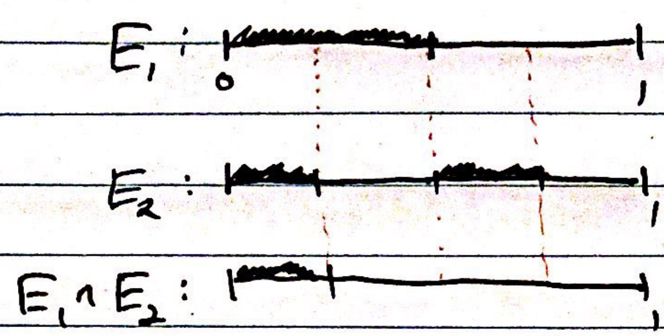
$$= \int_0^{1/4} dx + \int_{1/2}^{3/4} dx = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Independence: Two events E_1, E_2 are indep. if

$$p(E_1 \cap E_2) = p(E_1)\,p(E_2).$$

same as in discrete case

$$E_1 \cap E_2 = [0, 0.5] \cap \left([0, \tfrac{1}{4}] \cup [\tfrac{1}{2}, \tfrac{3}{4}]\right)$$



← first digit = 0

← second digit = 0

$= [0, \frac{1}{4}]$ ← first two = 0

$$p(E_1 \cap E_2) = \frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2} = p(E_1)\, p(E_2)$$

First two binary digits are like two independent coin flips.

· Every binary digit is like an independent coin flip, so we can think of the random number as being an infinite sequence of coin flips.

like ∞ly many indep. coin flips

In general, we'll consider distributions defined by a probability density function $p(x)$. The probability of an event ~~B~~ is given by

$$p(E) = \int_E p(x)\,dx .$$

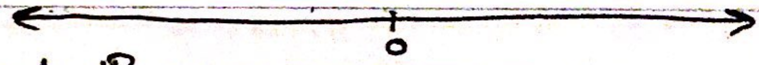
← must be positive and $\int p(x)\,dx =$

Outcomes: $[0, 1]$

Density function: $p(x) = 1$.

Gaussian / Normal Distribution

Outcomes: $\mathbb{R}$

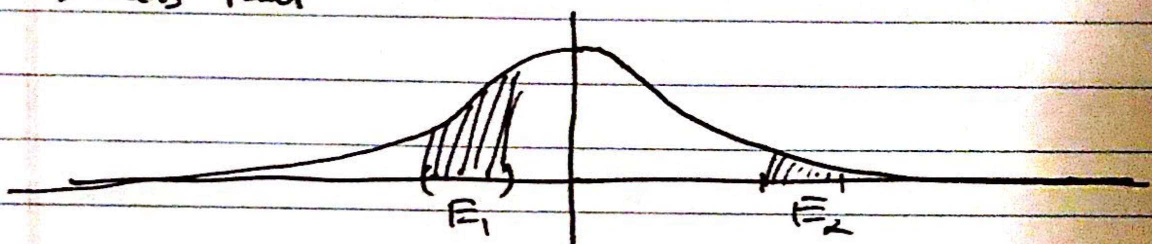


Events: Subsets of $\mathbb{R}$.

Density function: $p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$:

$$= \frac{1}{\sqrt{2\pi}} e^{-x^2/2} .$$

Means that



$$p(E_1) = \int_{E_1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \quad , \quad p(E_2) = \int_{E_2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

$$F(x) = \int_{-\infty}^{x} p(t)\,dt = p(t<x)$$ | CLT

cumulative dist. function

Random Variables, Mean, Variance

Recall:

- {Set of Outcomes, Ω
- Event: Subset of Outcomes
- Probability Distribution

<u>Defn</u>: A random variable X is a function $X : \Omega \to S$. Here $S = \mathbb{R}, \mathbb{R}^d$, but could be arbitrary.

<u>Ex</u>: Rolling a die:

- Outcomes $\Omega = \{1, 2, \ldots, 6\}$
- Events are subsets of Ω
- Distribution: $p(1) = p(2) = \cdots = p(6) = \frac{1}{6}$.

Suppose we roll the die, then if the die comes up d, you win d dollars, minus 1 dollar if it's even. The amount you win is a random variable

$$X_1 : \Omega \to \mathbb{R}$$

$$X_1(1) = 1$$
$$X_1(2) = 1$$
$$X_1(3) = 3$$
$$X_1(4) = 3$$
$$X_1(5) = 5$$
$$X_1(6) = 5 \, .$$

Can define multiple random variables on a single outcome space Ω

Ex: Rolling a die. If the die comes up d, you get d dollars. If d is even, then you give a dollar to your friend.

$X_1 \leftarrow$ your winnings.

$X_2 \leftarrow$ your friends winnings.

$$X_2 : \Omega \to \mathbb{R}, \quad X_2(1) = 0, \quad X_2(2) = 1,$$
$$X_2(3) = 0, \quad X_2(4) = 1,$$
$$X_2(5) = 0, \quad X_2(6) = 1$$

From here on out Ω will be fixed, we talk about different random variables on Ω.

Defn: Mean of a random variable X.

$$\mathbb{E}[X] = \cancel{\frac{1}{|\Omega|}} \sum_{\omega \in \Omega} p(\omega) X(\omega) \left(= \int_{\Omega} X(\omega) p(\omega) d\omega \right)$$

$\uparrow$ expectation of X

Ex: $\mathbb{E}[X_1] = \frac{1}{6}(1 + 1 + 3 + 3 + 5 + 5) = 3$

$\mathbb{E}[X_2] = \frac{1}{6}(0 + 1 + 0 + 1 + 0 + 1) = \frac{1}{2}.$

(real valued r.v.)

Variance of a Random Variable X:

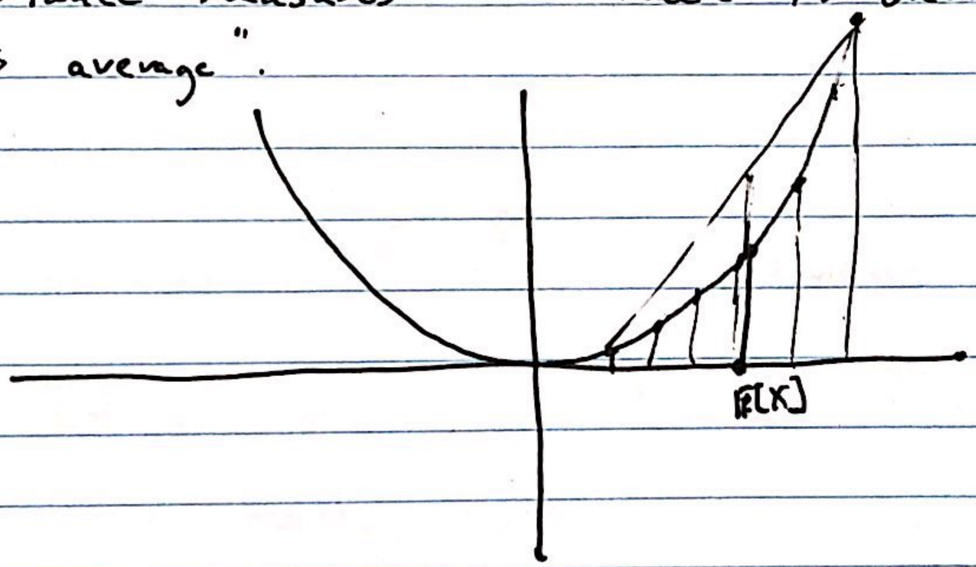
$$\mathbb{V}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \mathbb{E}[(X-\mathbb{E}[X])^2]$$

Ex: X_1: $\mathbb{V}[X_1] = \mathbb{E}[X_1^2] - (3)^2 = \frac{35}{3} - 9 = \frac{8}{3}$.

$\hookrightarrow \frac{1}{6}(1^2+1^2+3^2+3^2+5^2+5^2)$

$= \frac{1}{3}(1+9+25) = \frac{35}{3}$

Variance measures "how much X deviates from it's average".



Independence of Random variables:

Defn: Two Random variables X_1, X_2 are independent if for any α, β,

$$E_1 = \{\omega : X_1(\omega) < \alpha\},\ E_2 = \{\omega : X_2(\omega) < \beta\}$$

are independent events.

Ex: X_1, X_2 are independent:

(α, β) $\alpha = 4$, $\beta = \frac{1}{2}$.

$E_1 = \{w : X_1(w) < 4\} = \{1, 2, 3, 4\}$

$E_2 = \{w : X_2(w) < \frac{1}{2}\} = \{1, 3, 5\}$

$p(E_1 \cap E_2) = p(E_1)\, p(E_2)$

$$\underbrace{p(\{1,3\})}_{\frac{1}{3}} = \underbrace{p(E_1)}_{\frac{2}{3}}\, \underbrace{p(E_2)}_{\frac{1}{2}}$$

Properties of $\mathbb{E}$, $\mathbb{V}$, Independence:

- If X_1, X_2 are random variables, then $(a_1 X_1 + a_2 X_2)(w) = a_1 X_1(w) + a_2 X_2(w)$.
- $\mathbb{E}[a_1 X_1 + a_2 X_2] = a_1 \mathbb{E}[X_1] + a_2 \mathbb{E}[X_2]$
- If X_1, X_2 are independent random varicbles, then $\mathbb{E}[X_1 X_2] = \mathbb{E}[X_1]\mathbb{E}[X_2]$.
- From this, we get (X_1, X_2 indep.) $\mathbb{V}[X_1 + X_2] = \mathbb{V}[X_1] + \mathbb{V}[X_2]$
- $\mathbb{V}[a_1 X_1 + a_2 X_2] = a_1^2 \mathbb{V}[X_1] + a_2^2 \mathbb{V}[X_2]$.