HW Lesson 4: Convolutional Neural Networks

Problem 1 Consider the following multi-channel "image" (tensor)

$$g \in \mathbb{R}^{2\times4\times4}$$

where

$$[g]_1 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 2 & 2 & 2 & 2 \end{pmatrix} \quad [g]_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

We also consider the following convolution kernel with multichannel:

$$K \in \mathbb{R}^{2\times2\times3\times3},$$

where

$$K_{1,1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad K_{1,2} = \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{pmatrix}$$

$$K_{2,1} = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \qquad K_{2,2} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Please answer these following questions.

1. (**10%**) Compute the results for the following convolution operations for one channel with stride one and zero padding for $[g]_1$ and $[g]_2$:

$$[g]_1 * K_{1,1} \in \mathbb{R}^{4\times4} \quad \text{and} \quad [g]_2 * K_{2,2} \in \mathbb{R}^{4\times4}.$$

2. (**10%**) Compute the results for the following convolution operations for one channel with stride two and zero padding for $[g]_1$ and $[g]_2$:

$$[g]_1 *_2 K_{1,2} \in \mathbb{R}^{2\times2} \quad \text{and} \quad [g]_2 *_2 K_{2,1} \in \mathbb{R}^{2\times2}.$$

3. (**10%**) Compute the result for the following convolution operation for multi-channel with stride one and zero padding:

$$f = K * g \in \mathbb{R}^{2\times4\times4}.$$

Hint: Recall the definition of convolution with multi-channel that $[f]_i = \sum_{j=1}^{2} K_{i,j} * [g]_j$ for $i = 1, 2$.

4. (**10%**) Compute the result for the following convolution operation for multi-channel with stride two and zero padding:

$$\bar{f} = K *_2 g \in \mathbb{R}^{2\times2\times2}.$$

Hint: Recall the definition of convolution with multi-channel that $[\bar{f}]_i = \sum_{j=1}^{2} K_{i,j} *_2 [g]_j$ for $i = 1, 2$.

5. (**10%**) Consider max-pooling for g, please compute

$$R_{\max}(g),$$

with kernel size 2×2 and stride 2.

6. (**10%**) Consider $\mathrm{ReLU}(x) = \max\{0, x\}$, please compute

$$h = \mathrm{ReLU}\left(K * \mathrm{ReLU}(K * g)\right) \in \mathbb{R}^{2\times4\times4},$$

where $K*$ means the convolution for multi-channel with stride one and zero padding.

7. (**10%**) Consider $\mathrm{ReLU}(x) = \max\{0, x\}$, please compute

$$\bar{h} = \mathrm{ReLU}\left(K *_2 \mathrm{ReLU}(K * g)\right) \in \mathbb{R}^{2\times2\times2},$$

where $K*$ means the convolution using kernel K for multi-channel with stride one and zero padding, and $K*$ means the convolution using kernel K for multi-channel with stride two and zero padding.

Problem 2 (10 %) Consider another one channel "image"

$$g \in \mathbb{R}^{5\times5}$$

where

$$g = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let think about the 3×3 average kernel

$$R_{\text{ave}} = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Please calculate the following convolution for one channel with stride 2 with zero padding

$$f = R_{\text{ave}} *_2 g.$$

Problem 3 Consider 1D random variable X with an even probability density function $p(x)$ and finite variance, i.e., $p(x) = p(-x)$ and $\mathbb{V}[X] < \infty$. Define the normalization of X by

$$\tilde{X} = \frac{X - \mu}{s},$$

where

$$\mu = \mathbb{E}[X] = \int_{-\infty}^{\infty} xp(x)dx \quad \text{and} \quad s^2 = \mathbb{V}[X] = \mathbb{E}[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx.$$

1. (**15 %**) Prove that

$$\mathbb{E}[\tilde{X}] = 0 \quad \text{and} \quad \mathbb{V}[\tilde{X}] = 1.$$

2. (**5 %**) Consider $\text{ReLU}(x) = \max\{0, x\}$ and a new random variable Y defined by $Y = \text{ReLU}(\tilde{X})$, then prove that

$$\mathbb{E}[Y^2] = \frac{1}{2}.$$