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Robust block preconditioners for poroelasticity☆

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Abstract

In this paper we develop and analyze several preconditioners for the linear systems arising from discretized poroelasticity problems. The preconditioners include one block preconditioner for the two-field Biot model and several preconditioners for the classical three-field Biot model. We manage to analyze these different preconditioners under a same theoretical framework and show that all of them are uniformly optimal with respect to material and discretization parameters. Numerical tests demonstrate the robustness of these preconditioners.

Keywords: Poroelasticity; Biot model; Stable finite element method; Robust block preconditioners

1. Introduction

Poroelasticity, the study of the fluid flow in porous and elastic media, couples the elastic deformation with the fluid flow in porous media. As one popular poroelasticity model, the Biot model has wide applications in geoscience, biomechanics, and many other fields. There are many issues needed to be addressed in numerical simulations of poroelasticity such as the numerical instability of pressure variable under certain conditions [1–4]. One source of this instability is the instability of the finite element approximation for the coupled systems [2,3]. This motivates us to study the well-posedness of the finite element discretization.

Another interesting topic is the development of efficient linear solvers. Direct solvers have poor performance when the size of problems becomes large. Iterative solvers are good alternatives, as they exhibit better scalability, but the convergence of iterative solvers is known to be much problem-dependent such that there is a need for developing parameter-robust preconditioners. For example, the multigrid preconditioned Krylov subspace method usually has optimal convergence rate for the Poisson equation and many other symmetric positive definite problems [5,6]. However, for poroelasticity problems, coupled systems of equations must be solved, which are known to be indefinite and ill-conditioned [7]. Preconditioning techniques for poroelasticity problems have been the subject of considerable research in the literature [3,8–14] and most of the techniques developed are based on the Schur

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complement approach. In [10,11], diagonal approximation of the Schur complement preconditioner is used to precondition two-field formulation of the Biot model. In [12,13], Schur complement preconditioners are also studied for two-field formulation with the algebraic multigrid (AMG) as the preconditioner for the elasticity block. In [3], Schur complement approaches for three-field formulation are investigated. Recently, robust block diagonal and block triangular preconditioners are developed in [15] for two-field Biot model. And for classical three-field Biot model, the robust block preconditioners are designed in [16,17] based on the uniform stability estimates. Robust preconditioner for a new three-field formulation introducing a total pressure as the third unknown is analyzed in [18]. Robust block diagonal and block triangular preconditioners are also developed in [19] based on the discretization proposed in [20]. Other robustness analysis for fixed-stress splitting method and Uzawa-type method for multiple-permeability poroelasticity systems are presented in [21] and [22].

The focus of this paper is on the stability of the linear systems after time discretization and several robust preconditioners for the iterative solvers under the unified relationship framework between well-posedness and preconditioners. The block preconditioners in [15] for two field formulation and in [16,19] for the three field formulation can be briefly written in this framework. In addition, we analyze the well-posedness of the linear systems and propose other optimal preconditioners for the Biot model [2] based on the mapping property [23]. By proposing optimal block preconditioners, we convert the solution of complicated coupled system into that of a few symmetric positive definite (SPD) systems on each of the fields.

The rest of this paper is organized as follows. In Section 2, we give a brief introduction of the Biot model. In Section 3, we introduce one theorem in order to prove well-posedness. In Section 4, we address the unified framework indicating the relationship between preconditioning and well-posedness of linear systems. In Section 5 and Section 6, we show the well-posedness and several optimal preconditioners for the Biot model under the unified framework. In Section 7, we present numerical examples to demonstrate the robustness of these preconditioners.

2. The Biot model

The poroelastic phenomenon is usually characterized by the Biot model [24,25], which couples structure displacement $\mathbf{u}$, fluid flux $\mathbf{v}$, and fluid pressure p. Consider a bounded and simply connected Lipschitz domain $\Omega \subset \mathbb{R}^n (n = 2, 3)$ of poroelastic material. As the deformation is assumed to be small, we assume that the deformed configuration coincides with the undeformed reference configuration. Let σ denote the total stress in this material. From the balance of the forces, we first have

$$-\nabla \cdot \sigma = f, \quad \text{in } \Omega.$$

In addition to the elastic stress

$$\sigma_e = 2\mu\epsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I},$$

the fluid pressure also contributes to the total stress, which results in the following constitutive equation:

$$\sigma = \sigma_e - \alpha p\mathbf{I}.$$

Here, $\mu := \frac{E}{2(1+\nu)}$ and $\lambda := \frac{\nu E}{(1+\nu)(1-2\nu)}$ are the Lamé constants and $\nu \in [0, 1/2)$ is the Poisson ratio, the symmetric gradient is defined by $\epsilon(\mathbf{u}) := (\nabla\mathbf{u} + \nabla\mathbf{u}^T)/2$, and α is the Biot–Willis constant. Therefore, we obtain the following momentum equation

$$-\nabla \cdot (2\mu\epsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I} - \alpha p\mathbf{I}) = f, \quad \text{in } \Omega.$$

Let η denote the fluid content. Then, the mass conservation of the fluid phase implies that

$$\partial_t \eta + \nabla \cdot \mathbf{v} = g \quad \text{in } \Omega, \tag{1}$$

where g is the source density. The fluid content is assumed to satisfy the following constitutive equation:

$$\eta = Sp + \alpha\nabla \cdot \mathbf{u}, \tag{2}$$

where S is the fluid storage coefficient. We also have the Biot–Willis constant α in this equation, as this poroelastic model is assumed to be a reversible process and the increment of work must be an exact differential [24,26,27]. Based on (1) and (2), the following equation holds

$$\alpha\nabla \cdot \dot{\mathbf{u}} + \nabla \cdot \mathbf{v} + S\dot{p} = g, \quad \text{in } \Omega.$$

According to Darcy's law, we have another equation:

$$k^{-1}\mathbf{v} + \nabla p = r,$$

where k is the fluid mobility and r is the body force for the fluid phase.

We consider all the parameters to be positive. The following boundary conditions are assumed:

$$\mathbf{u} = \mathbf{u}_D, \text{ on } \Gamma_{D,u}, \quad \sigma\mathbf{n} = g_N, \text{ on } \Gamma_{N,u}, \tag{3}$$

$$\mathbf{v}\cdot\mathbf{n} = \mathbf{v}_D, \text{ on } \Gamma_{D,v}, \quad p = p_N, \text{ on } \Gamma_{N,v}, \tag{4}$$

where $\Gamma_{D,u} \cap \Gamma_{N,u} = \emptyset$, $\bar{\Gamma}_{D,u} \cup \bar{\Gamma}_{N,u} = \partial\Omega$ and $\Gamma_{D,v} \cap \Gamma_{N,v} = \emptyset$, $\bar{\Gamma}_{D,v} \cup \bar{\Gamma}_{N,v} = \partial\Omega$.

The initial conditions are as follows:

$$\mathbf{u}(x,0) = \mathbf{u}_0(x), \quad p(x,0) = p_0(x),$$

where $\mathbf{u}_0$ and p_0 are given functions.

We use the backward Euler method to discretize the time derivative $\dot{\mathbf{u}}$:

$$\dot{\mathbf{u}}(t_n) \approx \frac{\mathbf{u}(t_n) - \mathbf{u}(t_{n-1})}{\Delta t},$$

where Δt is the time step size. More sophisticated implicit time discretizations result in similar linear systems. As we are focusing on the properties of the linear systems resulting from the time discretized problem, we consider only the backward Euler method for the sake of brevity. After the implicit time discretization, fast solvers are needed to solve the following three-field system:

$$\begin{cases} -\nabla\cdot(2\mu\epsilon(\mathbf{u}) + \lambda(\nabla\cdot\mathbf{u})\mathbf{I}) + \alpha\nabla p = f, \\ k^{-1}\mathbf{v} + \nabla p = r, \\ \dfrac{\alpha}{\Delta t}\nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{v} + \dfrac{S}{\Delta t}p = g. \end{cases} \tag{5}$$

Note that the right-hand side of the last equation in (5) includes terms from previous time step due to the time discretization. As the exact form of this right-hand side does not affect the well-posedness of the linear system, we keep using g to denote it. We apply this convention to all the right-hand sides in similar situations, throughout the rest of this paper.

To reduce the number of variables, the fluid flux $\mathbf{v}$ is eliminated to obtain the following two-field system:

$$\begin{cases} -\nabla\cdot(2\mu\epsilon(\mathbf{u}) + \lambda(\nabla\cdot\mathbf{u})\mathbf{I}) + \alpha\nabla p = f, \\ \dfrac{\alpha}{\Delta t}\nabla\cdot\mathbf{u} - k\Delta p + \dfrac{S}{\Delta t}p = g. \end{cases} \tag{6}$$

In the rest of this paper, we develop block preconditioners for both the two-field and three-field systems.

3. Well-posedness of linear systems

In this section, we introduce the theorem to prove the well-posedness of the following saddle point problem: Find $(u,p) \in \mathbb{M}\times\mathbb{N}$ such that $\forall(\phi,q) \in \mathbb{M}\times\mathbb{N}$, the following equations hold

$$\begin{cases} a(u,\phi) + b(\phi,p) & = \langle f,\phi\rangle, \\ b(u,q) - c(p,q) & = \langle g,q\rangle. \end{cases} \tag{7}$$

Here, $\mathbb{M}$ and $\mathbb{N}$ are given Hilbert spaces with the inner products $(\cdot,\cdot)_{\mathbb{M}}$ and $(\cdot,\cdot)_{\mathbb{N}}$, respectively. The corresponding norms are denoted by $\|\cdot\|_{\mathbb{M}}$ and $\|\cdot\|_{\mathbb{N}}$.

Given $b(\cdot,\cdot)$, the following kernel spaces are important in the analysis:

$$\mathbb{Z} = \{u \in \mathbb{M} | b(u,q) = 0, \forall q \in \mathbb{N}\},$$

$$\mathbb{K} = \{p \in \mathbb{N} | b(\phi,p) = 0, \forall \phi \in \mathbb{M}\}.$$

We consider the orthogonal decompositions of $u \in \mathbb{M}$ and $p \in \mathbb{N}$ as follows:

$$u = u_0 + \bar{u}, \quad u_0 \in \mathbb{Z}, \ \bar{u} \in \mathbb{Z}^{\perp}, \quad p = p_0 + \bar{p}, \quad p_0 \in \mathbb{K}, \ \bar{p} \in \mathbb{K}^{\perp}.$$

We will use these notation to denote the components of functions in the kernel spaces and their orthogonal complements throughout the rest of this section.

The well-posedness of (7) can be proved provided that $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, and $c(\cdot,\cdot)$ satisfy certain properties.

Let $|\cdot|_e$ be a semi-norm on $\mathbb{N}$ such that $|p|_e \neq 0$, $\forall p \in \mathbb{K}^\perp$, $|p|_e = 0$ if $p \in \mathbb{K}$ and $\|q\|_{\mathbb{N}}^2 := |\bar{q}|_e^2 + |q|_c^2$, where $\bar{q} \in \mathbb{K}^\perp$ and $|q|_c^2 = c(q,q)$.

Remark 1. It is worth noting that in case $\mathbb{K} = \{0\}$, we have $\bar{q} = q$ and $\|q\|_{\mathbb{N}}^2 = |q|_e^2 + |q|_c^2$.

Assume that the following inequalities

$$a(u,\phi) \le C_a \|u\|_{\mathbb{M}} \|\phi\|_{\mathbb{M}}, \forall u, \phi \in \mathbb{M}, \tag{8}$$

$$b(u,p) \le C_b \|u\|_{\mathbb{M}} \|p\|_{\mathbb{N}}, \forall u \in \mathbb{M}, p \in \mathbb{N}, \tag{9}$$

$$c(p,q) \le C_c \|p\|_{\mathbb{N}} \|q\|_{\mathbb{N}}, \quad \forall p, q \in \mathbb{N}, \tag{10}$$

hold with the constants C_a, C_b, and C_c independent of parameters.

Theorem 1 ([28]). *Assume that $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric and positive semi-definite and that (8)–(10) hold. Moreover, assume that*

$$a(u,u) \ge \gamma_a \|u\|_{\mathbb{M}}^2, \quad \forall u \in \mathbb{Z}, \tag{11}$$

$$\sup_{u \in \mathbb{M}} \frac{b(u,q)}{\|u\|_{\mathbb{M}}} \ge \gamma_b \|q\|_{\mathbb{N}}, \forall q \in \mathbb{K}^\perp, \tag{12}$$

$$c(q,q) \ge \gamma_c \|q\|_{\mathbb{N}}^2, \quad \forall q \in \mathbb{K}, \tag{13}$$

where the constants γ_a, γ_b and γ_c are independent of the parameters. Then, Problem (7) is uniformly well-posed with respect to parameters under the norms $\|\cdot\|_{\mathbb{M}}$ and $\|\cdot\|_{\mathbb{N}}$.

Theorem 1 will be used to prove the well-posedness in different cases. Note that they are sufficient conditions for the problems to be well-posed. For weaker conditions, we refer to [29].

In this paper, we are especially interested in the robustness of preconditioners with respect to varying material and discretization parameters guided by the well-posedness of the linear system. Thus we want to emphasize the dependence on these parameters in inequalities. Therefore, we introduce the following notation: $\lesssim$, $\gtrsim$ and $\eqsim$. Given two quantities x and y, $x \lesssim y$ means that there is a constant C independent of these parameters such that $x \le Cy$. $\gtrsim$ can be similarly defined. $x \eqsim y$ if $x \lesssim y$ and $x \gtrsim y$.

4. Relationship between preconditioning and well-posedness

Given that a variational problem is well-posed, an optimal preconditioner can be developed, in order to speed up Krylov subspace methods, such as Conjugate Gradient Method (CG) and Minimal Residual Method (MINRES). In order to illustrate this fact, we first consider the following variational problem:

Find $\mathbf{x} \in \mathbb{X}$, such that

$$\mathbf{L}(\mathbf{x},\mathbf{y}) = \langle \mathbf{f}, \mathbf{y} \rangle, \quad \forall \mathbf{y} \in \mathbb{X}, \tag{14}$$

where $\mathbb{X}$ is a given Hilbert space and $\mathbf{f} \in \mathbb{X}'$.

The well-posedness of the variational problem (14) refers to the existence, uniqueness, and the stability $\|\mathbf{x}\|_{\mathbb{X}} \lesssim \|\mathbf{f}\|_{\mathbb{X}'}$ of the solution. The necessary and sufficient conditions for (14) to be well-posed are shown in the following theorem. We assume the symmetry $\mathbf{L}(\mathbf{x},\mathbf{y}) = \mathbf{L}(\mathbf{y},\mathbf{x})$ in the rest of this section.

Theorem 2 ([30]). *Problem (14) is well-posed if and only if the following conditions are satisfied:*

- *There exists a constant $C > 0$ such that $\mathbf{L}(\mathbf{x},\mathbf{y}) \le C\|\mathbf{x}\|_{\mathbb{X}}\|\mathbf{y}\|_{\mathbb{X}}$.*
- *There exists a constant $\beta_0 > 0$ such that*

$$\inf_{\mathbf{x} \in \mathbb{X}} \sup_{\mathbf{y} \in \mathbb{X}} \frac{\mathbf{L}(\mathbf{x},\mathbf{y})}{\|\mathbf{x}\|_{\mathbb{X}}\|\mathbf{y}\|_{\mathbb{X}}} = \beta_0 > 0. \tag{15}$$

Consider the operator form of (14):

$$\mathcal{L}\mathbf{x} = \mathbf{f} \in \mathbb{X}'.$$

Define operator $\mathcal{P}$ such that

$$(\mathcal{P}\mathbf{f}, \mathbf{y})_{\mathbb{X}} = \langle \mathbf{f}, \mathbf{y} \rangle, \quad \mathbf{f} \in \mathbb{X}', \mathbf{y} \in \mathbb{X}. \tag{16}$$

Assuming the well-posedness, then the following inequalities hold

$$\|\mathcal{P}\mathcal{L}\|_{\mathbf{L}(\mathbb{X},\mathbb{X})} = \sup_{\mathbf{x},\mathbf{y}} \frac{(\mathcal{P}\mathcal{L}\mathbf{x}, \mathbf{y})_{\mathbb{X}}}{\|\mathbf{x}\|_{\mathbb{X}}\|\mathbf{y}\|_{\mathbb{X}}} = \sup_{\mathbf{x},\mathbf{y}} \frac{\langle \mathcal{L}\mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|_{\mathbb{X}}\|\mathbf{y}\|_{\mathbb{X}}} \le C,$$

$$\|(\mathcal{P}\mathcal{L})^{-1}\|^{-1}_{\mathbf{L}(\mathbb{X},\mathbb{X})} = \inf_{\mathbf{x}} \sup_{\mathbf{y}} \frac{(\mathcal{P}\mathcal{L}\mathbf{x}, \mathbf{y})_{\mathbb{X}}}{\|\mathbf{x}\|_{\mathbb{X}}\|\mathbf{y}\|_{\mathbb{X}}} = \inf_{\mathbf{x}} \sup_{\mathbf{y}} \frac{\langle \mathcal{L}\mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\|_{\mathbb{X}}\|\mathbf{y}\|_{\mathbb{X}}} \ge \beta_0.$$

Therefore, the condition number of the preconditioned system is proved to be bounded

$$\hat{\kappa}(\mathcal{P}\mathcal{L}) := \|\mathcal{P}\mathcal{L}\|_{\mathbf{L}(\mathbb{X},\mathbb{X})}\|(\mathcal{P}\mathcal{L})^{-1}\|_{\mathbf{L}(\mathbb{X},\mathbb{X})} \le C/\beta_0.$$

This type of preconditioners is frequently used in the literature and is characterized as "mapping property" in a recent review paper [23].

Let $\{\boldsymbol{\phi}_i\}$ be a set of given basis of $\mathbb{X}$ and $\{\boldsymbol{\phi}'_i\}$ be a set of given basis of $\mathbb{X}'$. Consider the matrix representations of $\mathcal{P}$ and $\mathcal{L}$:

$$\mathcal{P}(\boldsymbol{\phi}'_1, \ldots, \boldsymbol{\phi}'_n) = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n)P, \quad \mathcal{L}(\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n) = (\boldsymbol{\phi}'_1, \ldots, \boldsymbol{\phi}'_n)L$$

and the vector representation of $\mathbf{x}$:

$$\mathbf{x} = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_n)x.$$

Assume L is symmetric and P is SPD. Denote the mass matrix of $\mathbb{X}$ by M, i.e., $M_{ij} = (\boldsymbol{\phi}_i, \boldsymbol{\phi}_j)_{\mathbb{X}}$, $\forall i, j$. In fact, $P = M^{-1}$. Then

$$\|\mathcal{P}\mathcal{L}\|_{\mathbf{L}(\mathbb{X},\mathbb{X})} = \sup_{\mathbf{x},\mathbf{y}} \frac{(\mathcal{P}\mathcal{L}\mathbf{x}, \mathbf{y})_{\mathbb{X}}}{\|\mathbf{x}\|_{\mathbb{X}}\|\mathbf{y}\|_{\mathbb{X}}} = \sup_{x,y} \frac{x^T(PL)^T My}{(x^T Mx)^{1/2}(y^T My)^{1/2}} = \max_{\lambda \in \sigma(PL)} |\lambda|.$$

Similarly,

$$\|(\mathcal{P}\mathcal{L})^{-1}\|^{-1}_{\mathbf{L}(\mathbb{X},\mathbb{X})} = \min_{\lambda \in \sigma(PL)} |\lambda|.$$

Therefore, $\hat{\kappa}(\mathcal{P}\mathcal{L}) = \hat{\kappa}(PL) = \frac{\max_{\lambda \in \sigma(PL)} |\lambda|}{\min_{\lambda \in \sigma(PL)} |\lambda|}$.

A more general approach is via norm equivalence matrices [31]. Given an SPD matrix H, H inner product and H norm can be defined correspondingly:

$$(x, x)_H := (Hx, x), \quad \|x\|_H^2 := (x, x)_H.$$

Nonsingular matrices A and B are *H-norm equivalent*, denoted by $A \sim_H B$, if there are constants γ and Γ independent of the size of the matrices such that

$$\gamma\|Bx\|_H \le \|Ax\|_H \le \Gamma\|Bx\|_H.$$

If $A \sim_H B$ and AB^{-1} is symmetric with respect to $(\cdot, \cdot)_H$, then MINRES preconditioned by B^{-1} has the following convergence estimate [31]:

$$\frac{\|r^k\|_H}{\|r^0\|_H} \le 2\left(\frac{\Gamma - \gamma}{\Gamma + \gamma}\right)^{k/2}.$$

Consider the preconditioner P defined as the matrix representation of $\mathcal{P}$ in (16). It is easy to see that $P^{-1} \sim_{M^{-1}} L$. Note that $P = M^{-1}$.

This can help in the design of preconditioners for CG and MINRES. Preconditioning GMRES differs in that it usually depends on the field of value analysis [31].

In the rest of the paper, we will use Theorem 1 to prove the well-posedness of the different formulations of the Biot model under different choices of $\mathbb{X}$. Then, based on the well-posedness, we show the corresponding optimal block preconditioners.

5. A two-field formulation

The preconditioning for the two-field system (6) has been studied extensively in the literature [10–13], where the Schur complement approach is usually used to develop preconditioners. In this paper, similar to [15], we briefly formulate a preconditioner based on the well-posedness of the linear systems for the two-field Biot model.

We first study the well-posedness of (6), beginning by changing the variable $\tilde{p} = -\alpha p$ in order to symmetrize (6). With an abuse of notation, we still use the notation p for pressure after the change of variable. Next, we introduce the function spaces for the displacement and the pressure. Due to the boundary conditions (3), we consider

$$\mathbb{U} \subset H_D^1(\Omega) := \{\mathbf{u} \in (H^1(\Omega))^n | \mathbf{u} = 0, \text{ on } \Gamma_{D,u}\}$$

for the displacement and

$$\mathbb{Q}_c \subset H_P^1(\Omega) := \{p \in H^1(\Omega) | p = 0, \text{ on } \Gamma_{N,v}\}$$

for the pressure. Here, we use the subscript "c" to suggest the continuity of the functions in $\mathbb{Q}_c$. We assume $|\Gamma_{D,u}| > 0$ in the rest of this paper so that the elasticity operator is nonsingular on $\mathbb{U}$. We also assume that $|\Gamma_{N,u}| > 0$ such that the divergence operator is surjective on the pressure space.

Let P_Q be the L^2 projection from $L^2(\Omega)$ to $\mathbb{Q}_c$ (space for pressure in two-field formulation case) or $\mathbb{Q}$ (space for pressure in three field formulation case). Then, we define the following bilinear forms:

$$\begin{aligned}
&\text{for } \mathbf{u}, \boldsymbol{\phi} \in \mathbb{U}, \quad a^I(\mathbf{u}, \boldsymbol{\phi}) = (2\mu\epsilon(\mathbf{u}), \epsilon(\boldsymbol{\phi})) + (\lambda P_Q \nabla \cdot \mathbf{u}, P_Q \nabla \cdot \boldsymbol{\phi}),\\
&\text{for } \mathbf{u} \in \mathbb{U}, \quad p \in \mathbb{Q}_c, \quad b^I(\mathbf{u}, p) = (\nabla \cdot \mathbf{u}, p),\\
&\text{for } p, q \in \mathbb{Q}_c, \quad d^I(p, q) = (\kappa^{-1}\nabla p, \nabla q) + (\xi p, q),
\end{aligned}$$

where $\kappa = \alpha^2/(\Delta t k)$ and $\xi = S/\alpha^2$.

Now, we introduce the notation for the kernel spaces:

$$\mathbb{Z}^I = \{\mathbf{u} \in \mathbb{U} | b^I(\mathbf{u}, q) = 0, \forall q \in \mathbb{Q}_c\}, \quad \mathbb{K}^I = \{p \in \mathbb{Q}_c | b^I(\boldsymbol{\phi}, p) = 0, \forall \boldsymbol{\phi} \in \mathbb{U}\}.$$

The variational formulation of (6) is as follows:

Find $(\mathbf{u}, p) \in \mathbb{U} \times \mathbb{Q}_c$ such that $\forall (\boldsymbol{\phi}, q) \in \mathbb{U} \times \mathbb{Q}_c$, the following equations hold

$$\begin{cases} a^I(\mathbf{u}, \boldsymbol{\phi}) + b^I(\boldsymbol{\phi}, p) &= (f, \boldsymbol{\phi}),\\ b^I(\mathbf{u}, q) - d^I(p, q) &= (g, q). \end{cases} \tag{17}$$

We define the norms as follows:

$$\|\mathbf{u}\|_{\mathbb{U}}^2 = a^I(\mathbf{u}, \mathbf{u}), \quad \|q\|_{\mathbb{Q}_c}^2 = \beta^{-1}\|q\|_0^2 + d^I(q, q), \tag{18}$$

where $\beta = \max\{\mu, \lambda\}$.

This variational formulation (17) is proved to be well-posed under the norms $\|\cdot\|_{\mathbb{U}}$ and $\|\cdot\|_{\mathbb{Q}_c}$ provided that the following inf-sup condition holds

$$\forall p \in (\mathbb{K}^I)^{\perp}, \quad \sup_{\mathbf{u} \in \mathbb{U}} \frac{b^I(\mathbf{u}, p)}{\|\mathbf{u}\|_1} \gtrsim \|p\|_0. \tag{19}$$

It is well known that (19) holds for $\mathbb{U} = H_D^1(\Omega)$, $\mathbb{Q} = L^2(\Omega)$ and $\mathbb{U} = (H_0^1(\Omega))^n$, $\mathbb{Q} = L_0^2(\Omega)$ on a bounded domain Ω with Lipschitz boundary [32,33]. Moreover, (19) holds for stable Stokes finite element method (FEM) pairs [29].

Theorem 3. *Assume that the inf-sup condition (19) holds and $\beta = \max\{\mu, \lambda\}$. The system (17) is uniformly well-posed with respect to parameters under the norms $\|\cdot\|_{\mathbb{U}}$ and $\|\cdot\|_{\mathbb{Q}_c}$ defined in (18).*

Proof. To prove the well-posedness, we just need to verify the assumptions of Theorem 1.

As we assume that $|\Gamma_{N,u}| > 0$, we know that $\mathbb{K}^I = \{0\}$ and then (13) is trivial. By definition, (8)–(11) are straightforward to verify.

Based on (19), the following inf-sup condition is implied

$$\forall p \in (\mathbb{K}^I)^{\perp}, \quad \sup_{\mathbf{u} \in \mathbb{U}} \frac{b^I(\mathbf{u}, p)}{\|\mathbf{u}\|_{\mathbb{U}}} \gtrsim \|p\|_{\mathbb{Q}}. \tag{20}$$

Then (12) is verified. Therefore, the proof is finished by applying Theorem 1.

In [15], a result similar to Theorem 3 for two-field formulation of Biot model is shown for a stabilized scheme, see Theorem 3 in [15]. Here we consider stable Stokes FEM pairs without stabilization and the proof is given under an abstract setting, namely Theorem 1.

With the well-posedness of (17) proved, an optimal preconditioner can be formulated. We first introduce some matrix notation. Given finite element basis functions $\{\mathbf{u}_i\}$ and $\{p_i\}$ for $\mathbb{U}$ and $\mathbb{Q}$, respectively, define the following stiffness matrices: $(A_u)_{ij} := a^I(\mathbf{u}_i, \mathbf{u}_j)$, $(B_u)_{ij} = b^I(\mathbf{u}_i, p_j)$, $(A_p)_{ij} = d^I(p_i, p_j)$ and $(M_p)_{ij} = (p_i, p_j)$.

The matrix forms of the system and preconditioner are

$$S^{II} = \begin{pmatrix} A_u & B_u^T \\ B_u & -A_p \end{pmatrix} \quad \text{and} \quad P^{II} = \begin{pmatrix} A_u & \\ & \beta^{-1} M_p + A_p \end{pmatrix}^{-1},$$

respectively.

Remark 2. In case $|\Gamma_{N,u}| = 0$, the kernel space $\mathbb{K}^I$ contains constant functions. We can similarly prove the well-posedness, but the norm $\|q\|_{\mathbb{N}}$ has a term $|\bar{q}|_e$, which results in a dense matrix in the preconditioner. We refer to [18] for constructing preconditioners related to $|\bar{q}|_e$.

In the literature, the preconditioners for two-field formulation are mostly based on Schur complement approaches. The exact Schur complement preconditioner of S^{II}, i.e.,

$$\begin{pmatrix} A_u & \\ & A_p + B_u A_u^{-1} B_u^T \end{pmatrix},$$

is known to be an optimal preconditioner [34], although $B_u A_u^{-1} B_u^T$ is dense and cannot be obtained. Practical approximations of $B_u A_u^{-1} B_u^T$, such as

$$B_u \mathrm{diag}(A_u)^{-1} B_u^T \quad \text{and} \quad \mathrm{diag}(B_u \mathrm{diag}(A_u)^{-1} B_u^T),$$

have also been investigated [10–13].

The two-field formulation is usually considered computationally efficient, as it involves the fewest variables and, therefore, has smaller linear systems to solve than the three-field formulation (5). However, the two-field formulation (with continuous pressure elements) exhibits oscillations in the pressure field, and more expanded systems such as the three-field formulation, are shown to be more stable [2,35]. Motivated by this fact, we study a three-field formulation [2] in the next section.

6. A three-field formulation

In this section, we will show the well-posedness of the three field formulation (5), briefly formulate the diagonal block robust preconditioners of [16,19] as special cases, and propose some new preconditioners for the three field formulation guided by the well-posedness.

6.1. The three-field formulation

We can write (5) as a symmetric problem by rescaling. Introduce

$$\tilde{\mathbf{v}} = \frac{\Delta t}{\alpha}\mathbf{v}, \quad \tilde{p} = -\alpha p.$$

The three-field system (5) can be rewritten as

$$\begin{cases} -\nabla \cdot (2\mu\epsilon(\mathbf{u}) + \lambda(\nabla \cdot \mathbf{u})\mathbf{I}) - \nabla \tilde{p} = f, \\ \kappa \tilde{\mathbf{v}} - \nabla \tilde{p} = r, \\ \nabla \cdot \mathbf{u} + \nabla \cdot \tilde{\mathbf{v}} - \xi \tilde{p} = g. \end{cases} \tag{21}$$

With an abuse of notation, we still use $\mathbf{v}$ and p to denote the scaled velocity $\tilde{\mathbf{v}}$, the scaled pressure $\tilde{p}$, respectively. Then, we introduce the function spaces:

$$\mathbb{V} \subset H_D(\mathrm{div}, \Omega) := \{\mathbf{v} \in H(\mathrm{div}, \Omega) | \mathbf{v} \cdot \mathbf{n} = 0, \ \text{on } \Gamma_{D,v}\},$$
$$\mathbb{W} = \mathbb{U} \times \mathbb{V}, \quad \mathbb{Q} \subset L^2(\Omega),$$

and bilinear forms

$$\begin{aligned}
&\text{for } (\mathbf{u}, \mathbf{v}), (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbb{W}, \quad a^{II}(\mathbf{u}, \mathbf{v}; \boldsymbol{\phi}, \boldsymbol{\psi}) = a^{I}(\mathbf{u}, \boldsymbol{\phi}) + (\kappa \mathbf{v}, \boldsymbol{\psi}),\\
&\text{for } (\mathbf{u}, \mathbf{v}) \in \mathbb{W}, \quad p \in \mathbb{Q}, \quad b^{II}(\mathbf{u}, \mathbf{v}; p) = b^{I}(\mathbf{u}, p) + (\nabla \cdot \mathbf{v}, p),\\
&\text{for } p, q \in \mathbb{Q}, \quad c^{I}(p, q) = (\xi p, q), \quad \xi > 0.
\end{aligned}$$

We define the corresponding kernel spaces related to $b^{II}(\cdot; \cdot)$

$$\begin{aligned}
\mathbb{Z}^{II} &= \{(\mathbf{u}, \mathbf{v}) \in \mathbb{W} | b^{II}(\mathbf{u}, \mathbf{v}; q) = 0, \forall q \in \mathbb{Q}\},\\
\mathbb{K}^{II} &= \{p \in \mathbb{Q} | b^{II}(\boldsymbol{\phi}, \boldsymbol{\psi}; p) = 0, \forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbb{W}\}.
\end{aligned}$$

Note that due to the assumption $|\Gamma_{N,u}| > 0$, we have $\mathbb{K}^{II} = \{0\}$.

Then, the weak formulation is as follows:

Find $(\mathbf{u}, \mathbf{v}) \in \mathbb{W}$ and $p \in \mathbb{Q}$ such that $\forall (\boldsymbol{\phi}, \boldsymbol{\psi}) \in \mathbb{W}$ and $q \in \mathbb{Q}$, the following equations hold

$$\begin{cases} a^{II}(\mathbf{u}, \mathbf{v}; \boldsymbol{\phi}, \boldsymbol{\psi}) + b^{II}(\boldsymbol{\phi}, \boldsymbol{\psi}; p) &= (f, \boldsymbol{\phi}) + (r, \boldsymbol{\psi}),\\ \quad b^{II}(\mathbf{u}, \mathbf{v}; q) - c^{I}(p, q) &= (g, q). \end{cases} \tag{22}$$

The additional term $c^{I}(p, q)$ corresponds to different versions of the Biot models [3].

The well-posedness of this saddle point problem can be proved with different choices of norms for $\mathbb{W}$ and $\mathbb{Q}$. We discuss some of these options in the rest of this section.

6.2. Augmented Lagrangian preconditioners

The stability of the three-field system (21) is closely related to the stability of the pairs $\mathbf{u}$–p and $\mathbf{v}$–p. In particular, it is considered stable if $\mathbf{u}$–p satisfies (19) and $\mathbf{v}$–p satisfies

$$\forall p \in (\mathbb{K}_v)^{\perp}, \qquad \sup_{\mathbf{v} \in \mathbb{V}} \frac{(\nabla \cdot \mathbf{v}, p)}{\|\mathbf{v}\|_{H(\mathrm{div})}} \gtrsim \|p\|_0, \tag{23}$$

where

$$\mathbb{K}_v := \{p \in \mathbb{Q} | (\nabla \cdot \mathbf{v}, p) = 0, \forall \mathbf{v} \in \mathbb{V}\}.$$

(23) holds for $\mathbb{V} = H_D(\mathrm{div}, \Omega)$ and $\mathbb{Q} = L^2(\Omega)$ and, in discrete cases, there are many stable pairs, such as Raviart–Thomas elements [36] for $\mathbb{V}$ and piecewise polynomials for $\mathbb{Q}$.

The augmented Lagrangian (AL) method [37,38] incorporates the constraint into the norm. The constraint here is

$$\nabla \cdot (\mathbf{u} + \mathbf{v}) = 0.$$

Therefore, it is natural to consider the following norms for the AL method.

We define the norms for spaces $\mathbb{W}$ and $\mathbb{Q}$ as follows:

$$\begin{aligned}
\|\mathbf{v}\|_{\mathbb{V}}^2 &= (\kappa \mathbf{v}, \mathbf{v}),\\
\|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}^2 &= \|\mathbf{u}\|_{\mathbb{U}}^2 + \|\mathbf{v}\|_{\mathbb{V}}^2 + \beta \|P_Q \nabla \cdot (\mathbf{u} + \mathbf{v})\|_0^2,\\
\|q\|_{\mathbb{Q}}^2 &= (\beta^{-1} q, q),
\end{aligned} \tag{24}$$

where ξ is the coefficient in bilinear form $c^{I}(\cdot, \cdot)$, and β is an undetermined parameter.

To prove the well-posedness of (22), we just need to verify the assumptions of Theorem 1.

Given (19), we have $\forall q \in (\mathbb{K}^{II})^{\perp}$,

$$\sup_{\mathbf{u}, \mathbf{v}} \frac{b^{II}(\mathbf{u}, \mathbf{v}; q)}{\|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}} \geq \sup_{\mathbf{u}} \frac{(\nabla \cdot \mathbf{u}, q)}{(\|\mathbf{u}\|_{\mathbb{U}}^2 + \beta \|P_Q \nabla \cdot \mathbf{u}\|_0^2)^{1/2}} \gtrsim \max\{\mu, \lambda, \beta\}^{-1/2} \|q\|_0. \tag{25}$$

For the case in which $\beta \geq \max\{\mu, \lambda\}$, the right-hand side of (25) is equal to $\|q\|_{\mathbb{Q}}$.

Given (23), we can prove another inequality: $\forall q \in (\mathbb{K}^{II})^{\perp}$

$$\sup_{\mathbf{u}, \mathbf{v}} \frac{b^{II}(\mathbf{u}, \mathbf{v}; q)}{\|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}} \geq \sup_{\mathbf{v}} \frac{(\nabla \cdot \mathbf{v}, q)}{(\|\mathbf{v}\|_{\mathbb{V}}^2 + \beta \|P_Q \nabla \cdot \mathbf{v}\|_0^2)^{1/2}} \gtrsim \max\{\kappa, \beta\}^{-1/2} \|q\|_0. \tag{26}$$

Similarly, if we further assume that $\beta \geq \kappa$, the right-hand side of (26) is equal to $\|q\|_{\mathbb{Q}}$. Note that this approach is used in [8], where the displacement $\mathbf{u}$ is set to be zero and the inf-sup condition of the $\mathbf{v}$–p pair is assumed.

The boundedness of $b^{II}(\cdot,\cdot)$ is easy to verify due to the additional term $\beta\|P_Q\nabla\cdot(\mathbf{u}+\mathbf{v})\|_0$ in the norm $\|\cdot\|_{\mathbb{W}}$:

$$b^{II}(\mathbf{u},\mathbf{v};q) \leq \|P_Q\nabla\cdot(\mathbf{u}+\mathbf{v})\|_0\|q\|_0 \leq \|(\mathbf{u},\mathbf{v})\|_{\mathbb{W}}\|q\|_{\mathbb{Q}}. \tag{27}$$

The coercivity of $a^{II}(\cdot,\cdot)$ is straightforward to prove, as

$$\forall(\mathbf{u},\mathbf{v}) \in \mathbb{Z}^{II}, \quad a^{II}(\mathbf{u},\mathbf{v};\mathbf{u},\mathbf{v}) \equiv \|(\mathbf{u},\mathbf{v})\|_{\mathbb{W}}^2. \tag{28}$$

Theorem 4. *Assume $\beta = \min\{\max\{\mu,\lambda\},\kappa\}$, $\xi\beta$ is uniformly bounded and the inf-sup conditions* (19) and (23) *hold. Then the system* (21) *is uniformly well-posed with respect to parameters under the norms $\|\cdot\|_{\mathbb{W}}$ and $\|\cdot\|_{\mathbb{Q}}$ defined in* (24).

Proof. As $\mathbb{K}^{II} = \{0\}$, (13) is trivial to prove. Consider q for the inf-sup condition of $b^{II}(\cdot,\cdot)$. Due to $\beta = \min\{\max\{\mu,\lambda\},\kappa\}$, the right-hand side of (25) or (26) is equal to $\|q\|_{\mathbb{Q}}$. Therefore, the inf-sup condition of $b^{II}(\cdot,\cdot)$ is proved.

As $0 < \xi \lesssim \beta^{-1}$, we can prove that $c^I(p,q) \lesssim \|p\|_{\mathbb{Q}}\|q\|_{\mathbb{Q}}$. Therefore, the assumptions of Theorem 1 hold. Then the proof is finished by applying Theorem 1.

It is obvious that we only need to assume either (19) or (23) to prove the well-posedness of (22).

Corollary 1. *Assume $\beta = \max\{\mu,\lambda\}$, $\xi\beta$ is uniformly bounded, and that the inf-sup condition* (19) *holds. The system* (21) *is uniformly well-posed with respect to parameters under the norms defined in* (24).

Proof. The proof follows from (25), (28), (9), and Theorem 1.

Corollary 2. *Assume that $\beta = \kappa$, $\xi\beta$ is uniformly bounded, and the inf-sup condition* (23) *holds. The system* (21) *is uniformly well-posed with respect to parameters under the norms defined in* (24).

Proof. The proof follows from (26), (28), (9), and Theorem 1.

Remark 3. The assumption that both (19) and (23) hold results in a smaller parameter β than the cases where only one of (19) and (23) holds.

Based on the well-posed formulation, we derive the corresponding optimal block diagonal preconditioner.

6.2.1. Matrix form

We introduce some additional matrix notation. Also, we introduce the FEM basis $\{\mathbf{v}_i\}$ for $\mathbb{V}$. Define the stiffness matrices $(M_v)_{ij} = (\mathbf{v}_i,\mathbf{v}_j)$, $(A_v)_{ij} = (\kappa\mathbf{v}_i,\mathbf{v}_j)$, $(C_p)_{ij} = c^I(p_i,p_j)$, and $(B_v)_{ij} = (\nabla\cdot\mathbf{v}_i, p_j)$.

Then the system matrix of the three-field formulation is

$$S^{III} = \begin{pmatrix} A_u & & B_u^T \\ & A_v & B_v^T \\ B_u & B_v & -C_p \end{pmatrix}.$$

The block preconditioner is

$$P_1^{III} = \begin{pmatrix} A_u + \beta B_u^T M_p^{-1} B_u & \beta B_u^T M_p^{-1} B_v & \\ \beta B_v^T M_p^{-1} B_u & A_v + \beta B_v^T M_p^{-1} B_v & \\ & & \beta^{-1} M_p + C_p \end{pmatrix}^{-1}.$$

In order to be uniformly optimal with respect to the parameters, β is chosen as follows:

- $\beta = \max\{\mu,\lambda\}$, if (19) holds;
- $\beta = \kappa$, if (23) holds;
- $\beta = \min\{\max\{\mu,\lambda\},\kappa\}$, if both (19) and (23) hold.

We note that in order to use P_1^{III}, we need to solve $\mathbf{u}, \mathbf{v}$ together. In fact, we can use block triangle or block diagonal preconditioners to solve the $\mathbf{u}, \mathbf{v}$ coupled subproblem. Then we can further obtain some preconditioners for the Biot model.

6.3. Block diagonal preconditioners

We can formulate block diagonal preconditioners based on (19). Define another pair of norms for spaces $\mathbb{W}$ and $\mathbb{Q}$:

$$\|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}^2 = \|\mathbf{u}\|_{\mathbb{U}}^2 + \|\mathbf{v}\|_{\mathbb{V}}^2 + \beta\|P_Q\nabla\cdot\mathbf{v}\|_0^2, \quad \|q\|_{\mathbb{Q}}^2 = \beta^{-1}\|q\|_0^2, \tag{29}$$

where $\beta = \min\{\max\{\mu, \lambda\}, \kappa\}$.

Theorem 5. *Assume $\beta = \min\{\max\{\mu, \lambda\}, \kappa\}$, $\xi\beta$ is uniformly bounded and the inf-sup conditions* (19) and (23) *hold. Then the system* (21) *is uniformly well-posed with respect to parameters under the norms $\|\cdot\|_{\mathbb{W}}$ and $\|\cdot\|_{\mathbb{Q}}$ defined in* (29).

Proof. We need to verify the assumptions of Theorem 1 in order to finish the proof. The inf-sup condition of $b^{II}(\cdot,\cdot)$ follows from (25) or (26) and the assumption that $\beta = \min\{\max\{\mu, \lambda\}, \kappa\}$. The boundedness of $b^{II}(\cdot,\cdot)$ can be shown to be uniform:

$$\begin{aligned} &b^{II}(\mathbf{u}, \mathbf{v}; p) \le (\|\nabla\cdot\mathbf{u}\|_0 + \|\nabla\cdot\mathbf{v}\|_0)\|p\|_0 \\ \lesssim &\|\mathbf{u}\|_{\mathbb{U}}\beta^{-1/2}\|p\|_0 + \|\mathbf{v}\|_{\mathbb{V}}\beta^{-1/2}\|p\|_0 \lesssim \|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}\|p\|_{\mathbb{Q}}. \end{aligned}$$

In the kernel $\mathbb{Z}^{II}$ we have $P_Q\nabla\cdot\mathbf{u} = -P_Q\nabla\cdot\mathbf{v}$; therefore, the coercivity can be shown as $\forall(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^{II}$

$$\begin{aligned} &a^{II}(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{v}) \gtrsim a^{II}(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{v}) + \beta\|\nabla\cdot\mathbf{u}\|_0^2 \\ \ge &a^{II}(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{v}) + \beta\|P_Q\nabla\cdot\mathbf{v}\|_0^2 = \|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}^2. \end{aligned}$$

The boundedness of $a^{II}(\cdot,\cdot)$ and the assumptions on $c^I(\cdot,\cdot)$ are straightforward to verify.

Corollary 3. *Assume that the inf-sup condition* (19) *holds, $\beta = \max\{\mu, \lambda\}$ and $\xi\beta$ is uniformly bounded. The system* (21) *is uniformly well-posed with respect to parameters under the norms $\|\cdot\|_{\mathbb{W}}$ and $\|\cdot\|_{\mathbb{Q}}$ defined in* (29).

Proof. The proof follows from Theorem 1 and the inf-sup condition of $b^{II}(\cdot,\cdot)$ follows from (25). And the boundedness of $b^{II}(\cdot,\cdot)$ and coercivity of $a^{II}(\cdot,\cdot)$ can be obtained similarly to the proof of Theorem 5.

Corollary 4. *Assume that $\beta = \kappa$, $\xi\beta$ is uniformly bounded, and the inf-sup condition* (23) *holds. The system* (21) *is uniformly well-posed with respect to parameters under the norms defined in* (29).

Proof. The proof follows from Theorem 1 and the inf-sup condition of $b^{II}(\cdot,\cdot)$ follows from (26). And the boundedness of $b^{II}(\cdot,\cdot)$ and coercivity of $a^{II}(\cdot,\cdot)$ can be obtained similarly to the proof of Theorem 5.

Remark 4. The assumption that both (19) and (23) hold results in a smaller parameter β than the cases where only one of (19) and (23) holds.

6.3.1. Matrix form

The matrix form of the block diagonal preconditioner is as follows:

$$P_2^{III} = \begin{pmatrix} A_u & & \\ & A_v + \beta B_v^T M_p^{-1} B_v & \\ & & \beta^{-1}M_p + C_p \end{pmatrix}^{-1},$$

where $\beta = \min\{\max\{\mu, \lambda\}, \kappa\}$.

The preconditioner P_2^{III} is closely related to the block diagonal preconditioner proposed in [16]. In fact, when we choose spaces $\mathbb{U}$ and $\mathbb{Q}$ such that $\nabla\cdot\mathbb{U} \subset \mathbb{Q}$ and $\nabla\cdot\mathbb{V} \subset \mathbb{Q}$, then the preconditioner P_2^{III} reduces to the preconditioner proposed in [16] with very small ξ and some rescaling of the parameters.

Table 1
Values of $\xi\beta$ for various poroelastic materials.

	$\xi\beta$		$\xi\beta$
Ruhr sandstone	2.3836	Tennessee marble	12.1667
Charcoal granite	6.7635	Berea sandstone	2.3192
Westerly granite	2.5972	Weber sandstone	2.9235
Ohio sandstone	3.5965	Pecos sandstone	2.5322
Boise sandstone	2.4860		

Table 2
Number of iterations of PMINRES with CG discretization for three-field scheme with $\kappa = 10^7$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	28	35	27	36	51	43	34	49	51
	1/32	33	47	37	37	55	53	33	51	55
	1/64	35	54	47	36	57	57	33	51	57
	1/128	36	56	55	36	58	60	33	51	59
0.49	1/16	29	40	29	30	42	37	28	37	35
	1/32	30	42	35	30	42	39	28	37	37
	1/64	30	44	39	30	42	39	26	37	37
	1/128	29	44	39	29	42	41	26	37	37
0.495	1/16	29	40	31	29	37	37	26	35	33
	1/32	29	42	35	28	38	37	26	35	35
	1/64	28	42	37	28	38	39	26	35	35
	1/128	28	42	39	28	38	39	26	35	35
0.499	1/16	27	39	33	26	35	33	23	30	31
	1/32	28	40	35	26	35	35	23	30	33
	1/644	27	40	35	26	35	35	23	30	33
	1/128	27	40	37	26	35	35	23	30	33

Compared to the block diagonal preconditioner proposed in [19], the parameter β in $H(\text{div})$ block of the preconditioner P_2^{III} will be smaller when κ is small. Namely the condition number of the $H(\text{div})$ problem $(\mathbf{v}, \boldsymbol{\psi}) + \beta(\nabla\cdot\mathbf{v}, \nabla\cdot\boldsymbol{\psi})$ will be smaller.

To obtain A_u^{-1} for elasticity subproblem, we can use the multigrid method proposed in [39] for the discontinuous Galerkin discretization which is robust with respect to the parameters μ and λ. In the previous approach, we added $\beta(\nabla\cdot\mathbf{v}, \nabla\cdot\mathbf{v})$ to the norm. This term causes some difficulty for the block solvers when β is large. There are lots of studies on this topic, and we may resort to Hiptmair–Xu preconditioners [40]. For $H(\text{div})$ problems with highly varying permeability, preconditioners based on additive Schur complement approximation proposed in [41] provide an alternative. Here, we can avoid this term by adding a Laplace-like term on the pressure diagonal block. This is also used in the mixed formulation for Poisson equations.

We define the norms for spaces $\mathbb{W}$ and $\mathbb{Q}$:

$$\|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}^2 = \|\mathbf{u}\|_{\mathbb{U}}^2 + \|\mathbf{v}\|_{\mathbb{V}}^2, \quad \|q\|_{\mathbb{Q}}^2 = \beta^{-1}\|q\|_0^2 + \|\text{div}_V^* q\|_{\mathbb{V}'}^2, \tag{30}$$

where $\beta = \max\{\mu, \lambda\}$ and $\text{div}_V^* : \mathbb{Q} \mapsto \mathbb{V}'$ is the adjoint operator of $\text{div}_V : \mathbb{V} \mapsto \mathbb{Q}'$; i.e.,

$$\langle \text{div}_V^* q, \mathbf{v}\rangle := \langle q, \text{div}_V \mathbf{v}\rangle, \quad \forall \mathbf{v} \in \mathbb{V}, q \in \mathbb{Q}.$$

Theorem 6. *Assume that the inf-sup conditions (19) and (23) hold and $\beta = \max\{\mu, \lambda\}$. The system (21) is uniformly well-posed with respect to parameters under the norms in (30).*

Proof. We use Theorem 1 to finish the proof.

First, we consider the inf-sup condition of $b^{II}(\cdot, \cdot)$.

Table 3
Number of iterations of PMINRES with CG discretization for three-field scheme with $\kappa = 10^5$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	34	49	51	29	40	41	24	33	35
	1/32	33	51	55	29	41	43	24	34	35
	1/64	33	51	57	29	41	43	24	34	35
	1/128	33	51	59	27	41	44	24	34	37
0.49	1/16	28	37	35	24	32	33	21	27	31
	1/32	28	37	37	24	32	35	21	28	33
	1/64	26	37	37	24	32	35	21	28	33
	1/128	26	37	37	24	32	35	21	28	33
0.495	1/16	26	35	33	22	30	33	21	26	31
	1/32	26	35	35	22	31	33	21	27	31
	1/64	26	35	35	22	31	33	21	27	31
	1/128	26	35	35	22	31	33	21	27	31
0.499	1/16	23	30	31	20	26	29	18	23	29
	1/32	23	30	33	20	26	31	19	23	29
	1/64	23	30	33	20	26	31	19	23	29
	1/128	23	30	33	20	26	31	19	23	29

Table 4
Number of iterations of PMINRES with CG discretization for three-field scheme with $\kappa = 10^3$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	24	33	35	21	28	33	20	25	31
	1/32	24	34	35	21	28	33	20	25	31
	1/64	24	34	35	21	28	33	20	25	31
	1/128	24	34	37	21	28	33	20	25	31
0.49	1/16	21	27	31	19	24	29	17	22	27
	1/32	21	28	33	19	24	31	17	22	29
	1/64	21	28	33	19	24	31	17	22	29
	1/128	21	28	33	19	24	31	17	22	29
0.495	1/16	21	26	31	19	23	29	17	21	27
	1/32	21	27	31	19	23	29	17	21	27
	1/64	21	27	31	19	23	29	17	21	27
	1/128	21	27	31	19	23	29	17	21	27
0.499	1/16	18	23	29	16	21	27	15	19	25
	1/32	19	23	29	16	21	27	15	19	25
	1/64	19	23	29	16	21	27	15	19	25
	1/128	19	23	29	16	21	27	15	19	25

Given that $q \in \mathbb{Q}$, we have the following inequalities:

$$\sup_{(\mathbf{u},\mathbf{v})} \frac{b^{II}(\mathbf{u},\mathbf{v};q)}{\|(\mathbf{u},\mathbf{v})\|_{\mathbb{W}}} \geq \sup_{\mathbf{u}} \frac{(\nabla\cdot\mathbf{u},q)}{\|\mathbf{u}\|_{\mathbb{U}}} \gtrsim \sup_{\mathbf{u}} \frac{(\nabla\cdot\mathbf{u},q)}{\beta^{1/2}\|\mathbf{u}\|_1} \gtrsim \beta^{-1/2}\|q\|_0,$$
$$\sup_{(\mathbf{u},\mathbf{v})} \frac{b^{II}(\mathbf{u},\mathbf{v};q)}{\|(\mathbf{u},\mathbf{v})\|_{\mathbb{W}}} \geq \sup_{\mathbf{v}} \frac{(\nabla\cdot\mathbf{v},q)}{\|\mathbf{v}\|_{\mathbb{V}}} = \|\mathrm{div}_V^* q\|_{\mathbb{V}'}.$$

Therefore, we have

$$\sup_{(\mathbf{u},\mathbf{v})} \frac{b^{II}(\mathbf{u},\mathbf{v};q)}{\|(\mathbf{u},\mathbf{v})\|_{\mathbb{W}}} \gtrsim \|q\|_{\mathbb{Q}}.$$

Table 5
Number of iterations of PMINRES with CG discretization for three-field scheme with $\kappa = 10$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	20	25	31	18	22	29	17	20	27
	1/32	20	25	31	18	22	29	17	20	27
	1/64	20	25	31	18	22	29	17	20	27
	1/128	20	25	31	18	22	29	18	20	27
0.49	1/16	17	22	27	16	19	27	14	17	25
	1/32	17	22	29	16	20	27	14	18	25
	1/64	17	22	29	16	20	27	14	17	25
	1/128	17	22	29	16	20	27	14	17	25
0.495	1/16	17	21	27	16	19	25	14	17	23
	1/32	17	21	27	16	19	25	14	17	23
	1/64	17	21	27	16	19	25	14	17	23
	1/128	17	21	27	16	19	25	14	17	23
0.499	1/16	15	19	25	13	17	23	13	16	21
	1/32	15	19	25	13	17	23	13	16	21
	1/64	15	19	25	13	17	23	13	16	21
	1/128	15	19	25	13	17	23	13	16	21

Table 6
Number of iterations of PMINRES with CG discretization for three-field scheme with $\kappa = 1$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	18	22	29	17	20	27	15	20	25
	1/32	18	22	29	17	20	27	15	20	25
	1/64	18	22	29	17	20	27	15	20	25
	1/128	18	22	29	17	20	27	15	20	25
0.49	1/16	16	20	27	14	17	25	13	17	23
	1/32	16	20	27	14	18	25	13	17	23
	1/64	16	20	27	14	17	25	13	17	23
	1/128	16	20	27	14	17	25	13	17	23
0.495	1/16	16	19	25	14	17	23	13	16	21
	1/32	16	19	25	14	17	23	13	16	21
	1/64	16	19	25	14	17	23	13	16	21
	1/128	16	19	25	14	17	23	13	16	21
0.499	1/16	13	17	23	13	16	21	12	14	19
	1/32	13	17	23	13	16	21	12	14	19
	1/64	13	17	23	13	16	21	12	14	19
	1/128	13	17	23	13	16	21	12	14	19

The boundedness of $b^{II}(\cdot, \cdot)$ can be shown to be uniform:

$$\begin{aligned} b^{II}(\mathbf{u}, \mathbf{v}; p) &= (\nabla \cdot \mathbf{u}, p) + (\nabla \cdot \mathbf{v}, p) \\ &\leq \|\mathbf{u}\|_{\mathbb{U}} \frac{1}{\beta^{1/2}} \|p\|_0 + \|\mathrm{div}^* p\|_{\mathbb{V}'} \|\mathbf{v}\|_{\mathbb{V}} \lesssim \|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}} \|p\|_{\mathbb{Q}}. \end{aligned}$$

$a^{II}(\cdot, \cdot)$ is coercive on $\mathbb{W}$ due to the fact that

$$a^{II}(\mathbf{u}, \mathbf{v}; \mathbf{u}, \mathbf{v}) \equiv \|(\mathbf{u}, \mathbf{v})\|_{\mathbb{W}}^2, \ \forall (\mathbf{u}, \mathbf{v}) \in \mathbb{W}.$$

The boundedness of $a^{II}(\cdot, \cdot)$ and the assumptions on $c^I(\cdot, \cdot)$ are straightforward to verify.

Table 7
Number of iterations of PMINRES with DG discretization for three-field scheme with $\kappa = 10^7$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	12	34	27	16	49	43	15	49	51
	1/32	14	46	37	16	55	53	15	51	51
	1/64	16	54	49	16	57	55	15	51	53
	1/128	16	58	55	16	58	57	15	51	57
0.49	1/16	10	40	29	10	41	37	16	37	35
	1/32	11	42	35	10	42	39	16	37	37
	1/64	12	44	39	10	42	41	16	37	37
	1/128	12	44	39	12	42	41	16	37	37
0.495	1/16	10	40	31	10	37	37	16	35	35
	1/32	10	42	35	10	37	37	16	35	35
	1/64	12	42	37	10	38	39	16	35	35
	1/128	12	42	39	11	38	39	16	35	35
0.499	1/16	9	38	33	15	35	33	17	30	31
	1/32	10	39	35	15	35	35	17	30	33
	1/64	11	40	37	15	35	35	17	30	33
	1/128	11	40	37	15	35	35	17	30	33

6.3.2. *Matrix form*

The block preconditioner is as follows:

$$P_3^{III} = \begin{pmatrix} A_u & & \\ & A_v & \\ & & \beta^{-1} M_p + \kappa^{-1} B_v M_v^{-1} B_v^T + C_p \end{pmatrix}^{-1},$$

where $\beta = \max\{\mu, \lambda\}$.

6.4. *Compare with Schur complement based preconditioners*

In [3], block preconditioners are proposed for the discretized Biot model of the following form:

$$\begin{pmatrix} A_u & & B_u^T \\ & A_v & B_v^T \\ B_u & B_v & -C \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ p \end{pmatrix} = \begin{pmatrix} f \\ r \\ g \end{pmatrix}, \tag{31}$$

where C is the pressure mass matrix M_p with constant coefficient. Block preconditioners for the case $C = M_p$ are proposed:

- pressure Schur complement:

$$P_{ps} = \begin{pmatrix} A_u & & \\ & A_v & \\ & & -C - B_v D_v^{-1} B_v^T \end{pmatrix}^{-1},$$

  where $-C - B_v D_v^{-1} B_v^T$ is shown in [8] to be spectrally equivalent to the exact Schur complement $-C - B_u A_u^{-1} B_u^T - B_v A_v^{-1} B_v^T$. This preconditioner is also used in [14].
- displacement–velocity Schur complement:

$$P_{uvs} = \begin{pmatrix} A_u + \beta B_u^T C^{-1} B_u & \beta B_u^T C^{-1} B_v & \\ \beta B_v^T C^{-1} B_u & A_v + \beta B_v^T C^{-1} B_v & \\ & & -C \end{pmatrix}^{-1}.$$

Table 8
Number of iterations of PMINRES with DG discretization for three-field scheme with $\kappa = 10^5$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	15	49	51	17	39	41	18	32	35
	1/32	15	51	51	17	39	43	18	33	37
	1/64	15	51	53	17	40	43	18	33	37
	1/128	15	51	59	17	41	45	18	33	37
0.49	1/16	16	37	35	17	32	33	17	27	31
	1/32	16	37	37	17	32	35	17	28	33
	1/64	16	37	37	17	32	35	17	28	33
	1/128	16	37	37	17	32	35	17	28	33
0.495	1/16	16	35	35	17	29	33	17	26	31
	1/32	16	35	35	17	31	33	17	27	31
	1/64	16	35	35	17	31	33	17	27	31
	1/128	16	35	35	17	31	33	17	27	31
0.499	1/16	17	30	31	16	26	31	16	23	29
	1/32	17	30	33	17	26	31	16	23	29
	1/64	17	30	33	17	26	31	16	23	29
	1/128	17	30	33	17	26	31	16	23	29

Table 9
Number of iterations of PMINRES with DG discretization for three-field scheme with $\kappa = 10^3$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	18	32	35	17	28	33	17	25	31
	1/32	18	33	37	18	28	33	17	25	31
	1/64	18	33	37	18	29	33	17	25	31
	1/128	18	33	37	18	29	33	17	25	31
0.49	1/16	17	27	31	16	24	31	16	22	29
	1/32	17	28	33	17	24	31	16	22	29
	1/64	17	28	33	17	24	31	16	22	29
	1/128	17	28	33	17	24	31	16	22	29
0.495	1/16	17	26	31	16	23	29	15	21	27
	1/32	17	27	31	16	23	29	15	21	27
	1/64	17	27	31	16	23	29	15	21	27
	1/128	17	27	31	16	23	29	15	21	27
0.499	1/16	16	23	29	15	21	27	14	19	25
	1/32	16	23	29	15	21	27	14	19	25
	1/64	16	23	29	15	21	27	14	19	25
	1/128	16	23	29	15	21	27	14	19	25

It is shown in [3] that exact solutions of the first 2-by-2 block of P_1^{III}, i.e.,

$$\begin{pmatrix} A_u + \beta B_u^T M_p^{-1} B_u & \beta B_u^T M_p^{-1} B_v \\ \beta B_v^T M_p^{-1} B_u & A_v + \beta B_v^T M_p^{-1} B_v \end{pmatrix}^{-1}, \tag{32}$$

result in uniform preconditioners. However, effective iterative solvers must be used for the inner iterations. In [3], two block preconditioners,

$$\begin{pmatrix} A_u & \\ & A_v \end{pmatrix}^{-1} \quad \text{and} \quad \begin{pmatrix} A_u + \beta B_u^T M_p^{-1} B_u & \\ & A_v + \beta B_v^T M_p^{-1} B_v \end{pmatrix}^{-1}, \tag{33}$$

are used to precondition (32). The numerical tests in [3] show that the second preconditioner in (33) results in a far fewer iterations for the inner iterative solvers than the first one. However, this approach introduces an additional

Table 10
Number of iterations of PMINRES with DG discretization for three-field scheme with $\kappa = 10$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/16	17	25	31	16	22	29	15	20	27
	1/32	17	25	31	16	22	29	16	20	27
	1/64	17	25	31	16	22	29	16	20	27
	1/128	17	25	31	18	22	29	16	20	27
0.49	1/16	16	22	29	15	20	27	14	17	25
	1/32	16	22	29	15	20	27	14	18	25
	1/64	16	22	29	15	20	27	14	18	25
	1/128	16	22	29	15	20	27	14	18	25
0.495	1/16	15	21	27	15	19	25	14	17	23
	1/32	15	21	27	15	19	25	14	17	23
	1/64	15	21	27	15	19	25	14	17	23
	1/128	15	21	27	15	19	25	14	17	23
0.499	1/16	14	19	25	13	16	23	12	16	21
	1/32	14	19	25	13	17	23	13	16	21
	1/64	14	19	25	13	17	23	13	16	21
	1/128	14	19	25	13	17	23	13	16	21

loop of iterative solvers. In [35], (32) is directly approximated by the second preconditioner in (33) and incomplete Cholesky factorization is used to further approximate the preconditioner.

The preconditioners, P_2^{III} and P_3^{III}, that we proposed are provably optimal and given their block diagonal forms that are easy to implement. Further, P_2^{III} and P_3^{III} have another advantage: they apply to the case where the diagonal block matrix $C = 0$ (i.e., the fluid storage coefficient S is zero), even though P_2^{III} is subject to the constraint $\xi \leq \beta^{-1}$.

6.5. *Values of $\xi\beta$ for various poroelastic materials*

Although some of the preconditioners we proposed depend on the assumption that $\xi\beta$ is uniformly bounded, $\xi\beta$ is usually small in various poroelastic materials. In Table 1, we calculate the corresponding values of $\xi\beta$ based on the poroelastic constants in [27].

7. Numerical tests

In 2D case, we test the preconditioners using the poroelastic footing experiment (see [4]). The domain is $\Omega = (-4, 4) \times (-4, 4)$. Define

$$\Gamma_1 = \{(x, y) \in \partial\Omega, |x| \leq 0.8, y = 4\}, \quad \Gamma_2 = \{(x, y) \in \partial\Omega, |x| > 0.8, y = 4\}.$$

The boundary conditions are as follows:

$$\begin{aligned} &(\sigma_e - p\mathbf{I})\mathbf{n} = -10^4, \mathbf{v} \cdot \mathbf{n} = 0, && \text{on } \Gamma_1, \\ &(\sigma_e - p\mathbf{I})\mathbf{n} = 0, p = 0, && \text{on } \Gamma_2, \\ &\mathbf{u} = 0, \mathbf{v} \cdot \mathbf{n} = 0, && \text{on } \partial\Omega/(\Gamma_1 \cup \Gamma_2). \end{aligned}$$

We assume that the fluid storage coefficient is $S = 0$ and the other material parameters are varying in huge range.

We discretize the problem using FEniCS [42]. We show the robustness of the preconditioners with respect to problem sizes and varying parameters. We discretize the problem on uniform triangular meshes. We use continuous Galerkin (CG) method with $P_2 \times RT_1 \times P_0$ and discontinuous Galerkin (DG) method with $BDM_1 \times RT_1 \times P_0$ for the three-field formulation [16]. Thus, the inf-sup conditions of $b^{II}(\cdot, \cdot)$ for both $\mathbf{u}$–p and $\mathbf{v}$–p are satisfied. We present the number of iterations of preconditioned MINRES (PMINRES) with the preconditioners for three-field formulation in Tables 2–6 (CG discretization) and Tables 7–11 (DG discretization), and elapsed solve times are

Table 11
Number of iterations and elapsed solve times of PMINRES with DG discretization for three-field scheme with $\kappa = 1$.

E	ν	h	P_1^{III}		P_2^{III}		P_3^{III}	
			Iteration	Time (s)	Iteration	Time (s)	Iteration	Time (s)
3×10^4	0.4	1/16	16	0.41	22	0.41	29	0.42
		1/32	16	0.85	22	0.82	29	0.84
		1/64	16	4.29	22	3.66	29	3.75
		1/128	16	31.57	22	24.27	29	24.95
	0.49	1/16	15	0.41	20	0.41	27	0.43
		1/32	15	0.85	20	0.80	27	0.86
		1/64	15	4.45	20	3.73	27	3.85
		1/128	15	31.71	20	24.01	27	23.58
	0.495	1/16	15	0.41	19	0.42	25	0.41
		1/32	15	0.85	19	0.80	25.	0.85
		1/64	15	4.27	19	3.67	25	3.73
		1/128	15	30.71	19	23.97	25	23.60
	0.499	1/16	13	0.41	16	0.43	23	0.42
		1/32	13	0.85	17	0.81	23	0.83
		1/64	13	4.16	17	3.57	23	3.65
		1/128	13	31.26	17	23.80	23	24.98
3×10^5	0.4	1/16	15	0.41	20	0.43	27	0.43
		1/32	16	0.85	20	0.83	27	0.85
		1/64	16	4.23	20	3.70	27	3.86
		1/128	16	31.10	20	24.96	27	24.37
	0.49	1/16	14	0.42	17	0.42	25	0.43
		1/32	14	0.87	18	0.82	25	0.85
		1/64	14	4.19	18	3.68	25	3.82
		1/128	14	33.73	18	23.96	25	24.41
	0.495	1/16	14	0.42	17	0.42	23	0.43
		1/32	14	0.88	17	0.82	23	0.83
		1/64	14	4.23	17	3.58	23	3.72
		1/128	14	30.36	17	24.70	23	24.22
	0.499	1/16	12	0.42	16	0.42	21	0.42
		1/32	13	0.87	16	0.82	21	0.82
		1/64	13	4.17	16	3.56	21	3.61
		1/128	13	30.95	16	24.72	21	23.50
3×10^6	0.4	1/16	15	0.42	19	0.41	25	0.42
		1/32	15	0.84	19	0.82	25	0.85
		1/64	15	4.22	20	3.64	25	3.65
		1/128	15	30.80	20	24.80	25	23.83
	0.49	1/16	13	0.41	16	0.42	23	0.42
		1/32	13	0.86	17	0.80	23	0.83
		1/64	13	4.16	17	3.69	23	3.72
		1/128	13	30.07	17	24.81	23	24.04
	0.495	1/16	13	0.41	16	0.41	21	0.41
		1/32	13	0.86	16	0.81	23	0.83
		1/64	13	4.22	16	3.61	23	3.64
		1/128	13	30.62	16	23.65	23	23.66
	0.499	1/16	12	0.42	13	0.42	19	0.42
		1/32	12	0.83	14	0.81	19	0.82
		1/64	12	4.12	14	3.68	19	3.54
		1/128	12	29.79	14	24.39	19	23.78

Table 12
Conditioned number of the unpreconditioned and preconditioned (with P_i^{III}, $i = 1, 2, 3$) system matrices on coarsest mesh.

	$E = 3 \times 10^4$			
ν	N/A	P_1^{III}	P_2^{III}	P_3^{III}
0.2	1.15×10^6	1.27	3.67	4.33
0.49	5.37×10^6	1.05	4.00	4.29
0.495	1.08×10^7	1.05	2.67	4.31
	$E = 3 \times 10^5$			
ν	N/A	P_1^{III}	P_2^{III}	P_3^{III}
0.2	1.15×10^7	1.03	3.68	4.30
0.49	5.36×10^7	1.01	4.00	4.30
0.495	1.08×10^8	1.01	2.67	4.30
	$E = 3 \times 10^6$			
ν	N/A	P_1^{III}	P_2^{III}	P_3^{III}
0.2	1.15×10^8	1.00	3.68	4.30
0.49	5.36×10^8	1.00	4.00	4.30
0.495	1.08×10^9	1.00	2.67	4.30

Table 13
Number of iterations of P_2^{III} with CG and DG discretization for three-field scheme with $\xi > \beta^{-1}$ ($\kappa = 10^7$).

ξ	ν	h	E					
			CG			DG		
			3×10^4	3×10^5	3×10^6	3×10^4	3×10^5	3×10^6
0.0001	0.4	1/16	16	26	52	16	26	51
		1/32	48	86	73	24	42	82
		1/64	32	55	107	32	57	111
		1/128	36	63	122	36	65	128
	0.499	1/16	44	45	39	43	45	39
		1/32	69	70	59	70	72	60
		1/64	92	92	75	95	96	84
		1/128	103	106	89	109	110	96
0.001	0.4	1/16	12	21	40	12	21	40
		1/32	18	35	70	18	37	71
		1/64	29	62	120	30	64	125
		1/128	45	96	184	46	101	198
	0.499	1/16	33	32	24	33	32	26
		1/32	55	53	40	58	57	42
		1/64	94	85	63	101	96	71
		1/128	139	127	94	158	150	110
0.1	0.4	1/16	7	11	19	7	11	19
		1/32	10	19	30	10	19	31
		1/64	17	33	50	18	35	55
		1/128	30	59	87	33	66	102
	0.499	1/16	10	4	2	11	4	2
		1/32	12	4	2	15	4	2
		1/64	15	4	2	23	5	2
		1/128	20	4	2	36	5	2

included in the case with $\kappa = 1$ in Table 11. For each of the preconditioners we showed, the number of iterations does not vary much with respect to the changing parameters and problem sizes. In addition, we also show the condition numbers of the unpreconditioned and preconditioned system matrices on the coarsest mesh (16×16) in

Table 14
Number of iterations of PMINRES with DG discretization for three-field scheme on three-dimensional domain with $\kappa = 10^7$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/8	13	35	27	16	53	45	16	55	63
	1/16	15	47	37	18	61	61	16	57	75
	1/32	17	59	53	18	64	77	16	59	99
0.49	1/8	11	38	27	12	42	43	16	39	45
	1/16	12	44	37	12	44	51	16	39	57
	1/32	12	46	47	12	44	69	16	39	77
0.495	1/8	11	38	31	11	40	43	16	37	43
	1/16	12	42	39	12	40	51	16	35	57
	1/32	12	44	47	12	40	69	16	35	75
0.499	1/8	11	38	37	15	35	41	17	33	41
	1/16	11	40	45	15	35	53	17	32	53
	1/32	11	40	61	15	35	71	17	32	69

Table 15
Number of iterations of PMINRES with DG discretization for three-field scheme on three-dimensional domain with $\kappa = 1$.

ν	h	$E = 3 \times 10^4$			$E = 3 \times 10^5$			$E = 3 \times 10^6$		
		P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}	P_1^{III}	P_2^{III}	P_3^{III}
0.4	1/8	16	22	35	16	20	33	15	18	31
	1/16	16	22	45	16	20	41	15	19	39
	1/32	16	22	61	16	20	55	15	19	51
0.49	1/8	15	20	33	14	18	31	14	17	29
	1/16	15	20	41	14	18	39	14	17	35
	1/32	15	20	55	14	18	51	14	17	45
0.495	1/8	15	19	31	14	17	29	13	16	27
	1/16	15	19	39	14	17	35	13	16	33
	1/32	15	19	53	14	17	49	13	16	43
0.499	1/8	14	17	29	13	16	27	12	14	23
	1/16	14	17	35	13	16	33	12	14	29
	1/32	14	17	45	13	16	43	12	14	35

Table 12. The condition numbers of the preconditioned systems are almost constant and close to 1. From Table 13, the numerical results show that although we need $\xi\beta$ to be uniform bounded in the theory, when $\xi\beta$ is bigger, the number of the iterations is getting much less, see the case $\xi = 0.1$ in Table 13.

In the following, we test numerical experiments on the three-dimensional domain, $\Omega = (-4, 4) \times (-4, 4) \times (-4, 4)$. Define

$$\Gamma_1 = \{(x, y, z) \in \partial\Omega, |x| \leq 0.8, |y| \leq 0.8, z = 4\},$$
$$\Gamma_2 = \{(x, y, z) \in \partial\Omega, |x| > 0.8, |y| > 0.8, z = 4\},$$

and the same boundary conditions as those in the two-dimensional problem are used. We present the numbers of iterations for three-dimensional examples in Tables 14–15.

8. Concluding remarks

In this paper we study the well-posedness of the linear systems arising from discretized poroelasticity problems. We formulate block preconditioner for the two-field Biot model and several preconditioners for the classical three-field Biot model under the unified relationship framework between well-posedness and preconditioners. By the unified theory, we show all the considered preconditioners are uniformly optimal with respect to material and discretization parameters. The preconditioners have block diagonal forms and reduce the global preconditioning to the local preconditioning. Numerical experiments have demonstrated the robustness of the preconditioners. Although we use the direct solvers for the analysis in this paper, we expect preconditioned iterative solvers for the local problems (like multigrid preconditioned MINRES) will result in robust iterative solvers for the whole systems. Although only block diagonal preconditioners are derived in this paper, these preconditioners can be also used to develop block triangular preconditioners [31].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

References

[1] M.A. Murad, A.F. Loula, On stability and convergence of finite element approximations of Biot's consolidation problem, Internat. J. Numer. Methods Engrg. 37 (4) (1994) 645–667.

[2] J.B. Haga, H. Osnes, H.P. Langtangen, On the causes of pressure oscillations in low-permeable and low-compressible porous media, Int. J. Numer. Anal. Methods Geomech. 36 (12) (2012) 1507–1522.

[3] O. Axelsson, R. Blaheta, P. Byczanski, Stable discretization of poroelasticity problems and efficient preconditioners for arising saddle point type matrices, Comput. Vis. Sci. 15 (4) (2012) 191–207.

[4] G. Aguilar, F. Gaspar, F. Lisbona, C. Rodrigo, Numerical stabilization of Biot's consolidation model by a perturbation on the flow equation, Internat. J. Numer. Methods Engrg. 75 (11) (2008) 1282–1300.

[5] W. Hackbusch, Multi-Grid Methods and Applications, Vol. 4, Springer-Verlag Berlin, 1985.

[6] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Rev. 34 (4) (1992) 581–613.

[7] M. Ferronato, G. Gambolati, P. Teatini, Ill-conditioning of finite element poroelasticity equations, Int. J. Solids Struct. 38 (34) (2001) 5995–6014.

[8] K. Lipnikov, Numerical Methods for the Biot Model in Poroelasticity (Ph.D. thesis), University of Houston, 2002.

[9] L. Bergamaschi, M. Ferronato, G. Gambolati, Novel preconditioners for the iterative solution to FE-discretized coupled consolidation equations, Comput. Methods Appl. Mech. Engrg. 196 (25) (2007) 2647–2656.

[10] K. Phoon, K. Toh, S. Chan, F. Lee, An efficient diagonal preconditioner for finite element solution of Biot's consolidation equations, Internat. J. Numer. Methods Engrg. 55 (4) (2002) 377–400.

[11] K.-C. Toh, K.-K. Phoon, S.-H. Chan, Block preconditioners for symmetric indefinite linear systems, Internat. J. Numer. Methods Engrg. 60 (8) (2004) 1361–1381.

[12] J.B. Haga, H. Osnes, H.P. Langtangen, Efficient block preconditioners for the coupled equations of pressure and deformation in highly discontinuous media, Int. J. Numer. Anal. Methods Geomech. 35 (13) (2011) 1466–1482.

[13] J.B. Haga, H. Osnes, H.P. Langtangen, A parallel block preconditioner for large-scale poroelasticity with highly heterogeneous material parameters, Comput. Geosci. 16 (3) (2012) 723–734.

[14] E. Turan, P. Arbenz, Large scale micro finite element analysis of 3D bone poroelasticity, Parallel Comput. 40 (7) (2014) 239–250, http://dx.doi.org/10.1016/j.parco.2013.09.002, URL http://www.sciencedirect.com/science/article/pii/S0167819113001063, 7th Workshop on Parallel Matrix Algorithms and Applications.

[15] J.H. Adler, F.J. Gaspar, X. Hu, C. Rodrigo, L.T. Zikatanov, Robust block preconditioners for Biot's model, in: International Conference on Domain Decomposition Methods, Springer, 2017, pp. 3–16.

[16] Q. Hong, J. Kraus, Parameter-robust stability of classical three-field formulation of Biot's consolidation model, Electron. Trans. Numer. Anal. 48 (2018) 202–227.

[17] Q. Hong, J. Kraus, M. Lymbery, F. Philo, Conservative discretizations and parameter-robust preconditioners for Biot and multiple-network flux-based poroelasticity models, Numer. Linear Algebra Appl. (2018) e2242.

[18] J.J. Lee, K.-A. Mardal, R. Winther, Parameter-robust discretization and preconditioning of Biot's consolidation model, SIAM J. Sci. Comput. 39 (1) (2017) A1–A24.

[19] J.H. Adler, F.J. Gaspar, X. Hu, P. Ohm, C. Rodrigo, L.T. Zikatanov, Robust preconditioners for a new stabilized discretization of the poroelastic equations, 2019, arXiv preprint arXiv:1905.10353.

[20] C. Rodrigo, X. Hu, P. Ohm, J.H. Adler, F.J. Gaspar, L.T. Zikatanov, New stabilized discretizations for poroelasticity and the Stokes equations, Comput. Methods Appl. Mech. Engrg. 341 (2018) 467–484.

[21] Q. Hong, J. Kraus, M. Lymbery, M.F. Wheeler, Parameter-robust convergence analysis of fixed-stress split iterative method for multiple-permeability poroelasticity systems, 2018, arXiv preprint arXiv:1812.11809.

[22] Q. Hong, J. Kraus, M. Lymbery, F. Philo, Parameter-robust Uzawa-type iterative methods for double saddle point problems arising in Biot's consolidation and multiple-network poroelasticity models, 2019, arXiv preprint arXiv:1910.05883.

[23] K.-A. Mardal, R. Winther, Preconditioning discretizations of systems of partial differential equations, Numer. Linear Algebra Appl. 18 (1) (2011) 1–40.

[24] M.A. Biot, General theory of three-dimensional consolidation, J. Appl. Phys. 12 (2) (1941) 155–164.

[25] M.A. Biot, Theory of elasticity and consolidation for a porous anisotropic solid, J. Appl. Phys. 26 (2) (1955) 182–185.

[26] J. Rice, Elasticity of fluid-infiltrated porous solids, Handout Adv. Environ. Geomech. (2001).

[27] A.H.-D. Cheng, Fundamentals of poroelasticity, Anal. Des. Methods: Compr. Rock Eng.: Princ. Pract. Proj. (2014) 113.

[28] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag New York, 1991.

[29] D. Boffi, F. Brezzi, M. Fortin, Mixed Finite Element Methods and Applications, in: Springer Series in Computational Mathematics, vol. 44, Springer Berlin Heidelberg, 2013.

[30] I. Babuška, Error-bounds for finite element method, Numer. Math. 16 (4) (1971) 322–333.

[31] D. Loghin, A.J. Wathen, Analysis of preconditioners for saddle-point problems, SIAM J. Sci. Comput. 25 (6) (2004) 2029–2049.

[32] J.H. Bramble, R.D. Lazarov, J.E. Pasciak, Least-squares methods for linear elasticity based on a discrete minus one inner product, Comput. Methods Appl. Mech. Engrg. 191 (8) (2001) 727–744.

[33] J.H. Bramble, A proof of the inf–sup condition for the Stokes equations on Lipschitz domains, Math. Models Methods Appl. Sci. 13 (03) (2003) 361–371.

[34] H. Elman, D. Silvester, A. Wathen, Finite Elements and Fast Iterative Solvers: With Applications in Incompressible Fluid Dynamics, Oxford University Press, USA, 2005, pp. 9–67, Secretary.

[35] M. Ferronato, N. Castelletto, G. Gambolati, A fully coupled 3-D mixed finite element model of Biot consolidation, J. Comput. Phys. 229 (12) (2010) 4813–4830.

[36] P.-A. Raviart, J.-M. Thomas, A mixed finite element method for 2-nd order elliptic problems, in: Mathematical Aspects of Finite Element Methods, Springer, 1977, pp. 292–315.

[37] M. Benzi, M.A. Olshanskii, An augmented Lagrangian-based approach to the Oseen problem, SIAM J. Sci. Comput. 28 (6) (2006) 2095–2113.

[38] J. Xu, K. Yang, Well-posedness and robust preconditioners for discretized fluid–structure interaction systems, Comput. Methods Appl. Mech. Engrg. 292 (2015) 69–91, http://dx.doi.org/10.1016/j.cma.2014.09.034, Special Issue on Advances in Simulations of Subsurface Flow and Transport (Honoring Professor Mary F. Wheeler).

[39] Q. Hong, J. Kraus, J. Xu, L. Zikatanov, A robust multigrid method for discontinuous Galerkin discretizations of Stokes and linear elasticity equations, Numer. Math. 132 (1) (2016) 23–49.

[40] R. Hiptmair, J. Xu, Nodal auxiliary space preconditioning in **H**(curl) and **H**(div) spaces, SIAM J. Numer. Anal. 45 (6) (2007) 2483–2509, http://dx.doi.org/10.1137/060660588, (electronic).

[41] J. Kraus, R. Lazarov, M. Lymbery, S. Margenov, L. Zikatanov, Preconditioning heterogeneous H(div) problems by additive schur complement approximation and applications, SIAM J. Sci. Comput. 38 (2) (2016) A875–A898.

[42] A. Logg, K.-A. Mardal, G.N. Wells, et al., Automated Solution of Differential Equations by the Finite Element Method, in: Lecture Notes in Computational Science and Engineering, vol. 84, Springer Berlin Heidelberg, 2012.